Non-Commutative Rational Functions in Random Matrices and Operators

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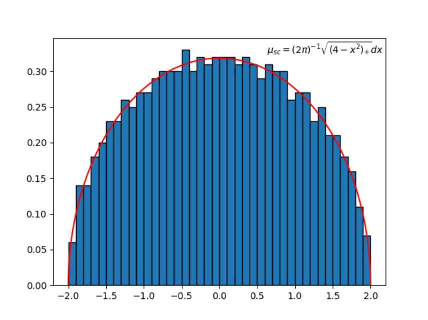
Outline of the Talk

1. Motivation: Inverse of a Wigner Matrix
2. Non-Commutative Rational Functions
3. Rational Functions in Strongly Convergent Random Matrices
4. Rational Functions in (Unbounded) Operators

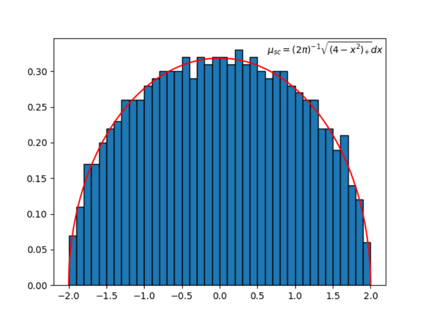
The inverses of random matrices

Let X be a GUE matrix and Y a Wigner matrix with entries in Bernoulli distributions with values 0 and 1

Eigenvalues of X



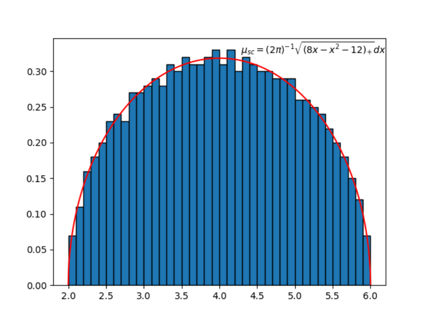
Eigenvalues of Y



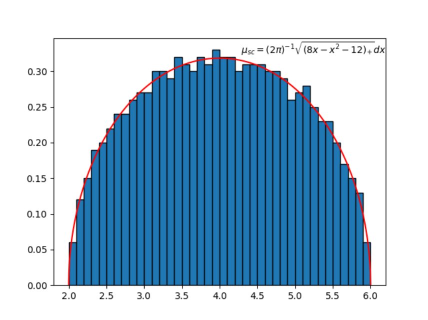
The inverses of random matrices

Let X be a GUE matrix and Y a Wigner matrix with entries in Bernoulli distributions with values 0 and 1

Eigenvalues of 4-X



Eigenvalues of 4-Y



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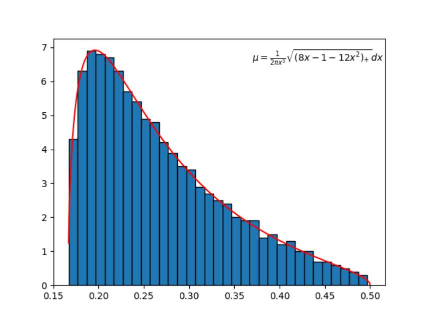
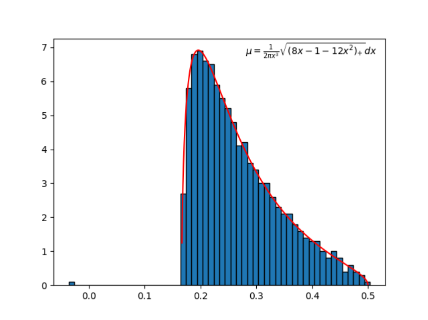
Eigenvalues of $(4 - X)^{-1}$



Eigenvalues of $(4 - Y)^{-1}$



The inverses of random matrices

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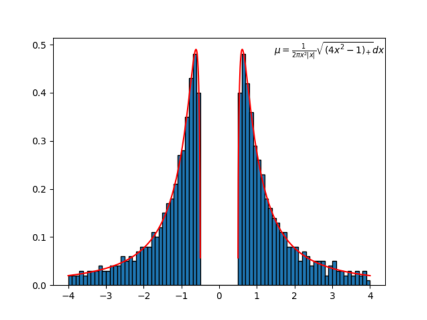
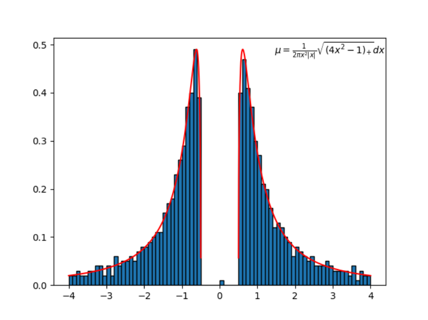
Eigenvalues of X^{-1}



Eigenvalues of Y^{-1}



The inverse of random matrices

Observation

A “nice” random matrix and a “nice” operator behave nicely with respect to taking inverse.

Questions

- What will happen if there are two or more variables?
- What are “nice ” random matrix?
- What are “nice” operators?

Non-commutative rational functions

What are rational functions?

- Roughly speaking, a rational function should be a equivalent class of expressions which make sense, like
  $(x^{-1}+y^{-1}+z^{-1})^{-1} = z(z+xy^{-1}z+x)^{-1}x$,
  $(x-y^{-1})^{-1} = x^{-1}+(xyx-x)^{-1}$...
- Different from one variable case (or commutative case), two polynomials are not enough to represent a rational function in general. For example, $(x^{-1}+y^{-1}+z^{-1})^{-1}$.

Non-commutative rational functions

Non-commutative rational functions (Amitsur 1966, Cohn 1971)

There exists a unique universal smallest skew field (or division ring) containing the ring of non-commutative polynomials, which is called the *free field*.

Remarks

- Roughly speaking, a rational function needs a matrix of polynomials to represent.
- In the commutative case, an integral domain always can be embedded into a smallest field, namely, the field of fractions. However, without universality, this embedding may not be unique any more for non-commutative case.

Evaluation of rational functions

Evaluation problem

Different expressions of a rational function should give same evaluation.

A bad example of evaluation: $r = y(xy)^{-1}x = 1$

Let $\mathcal{A} = B(H)$ for some Hilbert space with a basis $\{e_i\}_{i=1}^{\infty}$. Consider the one-sided left-shift operator l, then we have the property $ll^* = 1$ but $l^*l \neq 1$. For expression $y(xy)^{-1}x$, its evaluation at (l, l^*) is

$$r(l, l^*) = l^*(ll^*)l = l^*l \neq 1.$$

However, on the other hand, as a rational function $r = 1$, $r(l, l^*) = 1$. Algebras like B(H) are too large to define the evaluation of rational functions.

Evaluation of rational functions

Definition

A unital algebra $\mathcal{A}$ is stably finite (aka weakly finite) if for each $n \in M_n(\mathcal{A})$, any $A, B \in M_n(\mathcal{A})$, we have that $AB = 1$ implies $BA = 1$.

Theorem (Cohn)

$\mathcal{A}$ is stable finite if and only if all rational expressions of the zero function have zero evaluation.

Remarks

Any C^*-probability space with a faithful trace is stable finite.
In particular, $M_n(\mathbb{C})$ is stable finite. And the C^* algebra generated by $l + l^*$ is stable finite.

Recursive structure of rational functions

Let $\mathcal{P}$ be the ring of all non-commutative polynomials and $\mathcal{R}$ the free field. We can define $\mathcal{R}_1$ as the subring of $\mathcal{R}$ generated by the sets $\mathcal{P}$ and $\mathcal{P}^{-1} := \{p^{-1} | p \neq 0\}$. Recursively, then we can define $\mathcal{R}_n$ as the subring generated by $\mathcal{R}_{n-1}$ and $\mathcal{R}_{n-1}^{-1}$.

Recursive structure and height

- We have
$$\mathcal{R} = \bigcup_{n=1}^{\infty} \mathcal{R}_n.$$
- For any rational function $r \in \mathcal{R}$, there exists $n \in \mathbb{N}$ s.t. $r \in R_n$ and $r \notin \mathcal{R}_{n-1}$, called the height of r.
- Basically, the height means that how many nested inversions we need to construct, for example,
$$r = (x - y^{-1})^{-1} = x^{-1} + (xyx - x)^{-1} \in \mathcal{R}_1.$$

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Recursive structure and height

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- For any rational function $r \in \mathcal{R}$, there exists $n \in \mathbb{N}$ s.t. $r \in R_n$ and $r \notin \mathcal{R}_{n-1}$, called the height of r.
- For the commutative case, $\mathcal{R} = \mathcal{R}_1$ since each rational function can be written as a fraction of two polynomials. But in general, it is not the case, for example, $(x^{-1} + y^{-1} + z^{-1})^{-1} \in \mathcal{R}_2 \backslash \mathcal{R}_1$.

Asymptotic freeness of random matrices

Definition

Let $(\mathcal{A}, \tau)$ be some C*-probability spaces with faithful trace τ. We say a sequence of tuples of matrices $X^N = (X_1^N, \cdots, x_m^N) \in M_N(\mathbb{C})^m$ converges in distribution to a tuple $x = (x_1, \cdots, x_m) \in \mathcal{A}^m$ if

$$\lim_{N\to\infty} \mathrm{tr}_N \left(p(X^N, (X^N)^*) \right) = \tau(p(x, x^*))$$

for any polynomial p, where tr_N is the normalized trace on $M_N(\mathbb{C})$.

Example

If $(X_1^{(N)}, \cdots, X_m^{(N)})$ is a tuple of independent $N \times N$ GUE random matrices, then almost surely,

$$\lim_{N\to\infty} \mathrm{tr}_N \left(p(X_1^N, \cdots, X_m^N) \right) = \tau(p(s_1, \cdots, s_m))$$

for any p, where $s_1, \cdots, s_m$ are freely independent semi-circular elements.

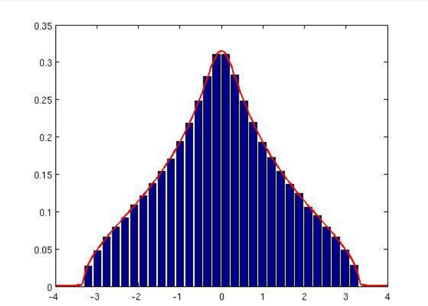
Limiting distribution of independent random matrices

Theorem (S. Belinschi, T. Mai, R. Speicher, 2013)

Let x be a tuple of self-adjoint freely independent operators. Then for any self-adjoint polynomial p, there is a general algorithm to calculate the distribution of $p(x)$ from the distributions of elements in x.

Example

Let s_1 and s_2 be two free semi-circular operators, and
$p(s_1, s_2) = s_1 s_2 + s_2 s_1$.



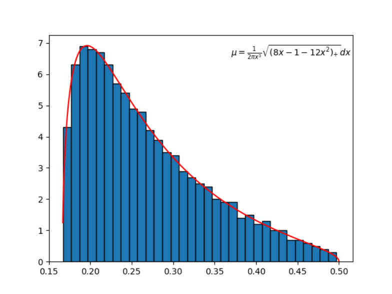
Limiting distribution of independent random matrices

Theorem (J. Helton, T. Mai and R. Speicher, 2015)

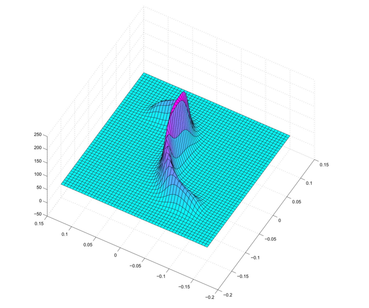
Let x be a tuple of self-adjoint freely independent operators. Then for any rational function r, there is a general algorithm to calculate the Brown measure of $r(x)$ from the distributions of elements in x.

Let x and y be two free semi-circular operators

$(4-x)^{-1}$



$-(4-x)^{-1}y[4-x-4iy(4-x)^{-1}y]^{-1}$



Strongly asymptotic freeness of random matrices

Definition

A sequence of tuples $X^N = (X_1^N, \cdots, x_m^N) \in M_N(\mathbb{C})^m$ converges strongly in distribution to a tuple $x = (x_1, \cdots, x_m)$ of operators if for any polynomial p,

- $\lim_{N\to\infty} \mathrm{tr}_N \left(p(X^N, (X^N)^*) \right) = \tau(p(x, x^*))$,
- $\lim_{N\to\infty} \left\| p(X^N, (X^N)^*) \right\|_{M_N(\mathbb{C})} = \|p(x, x^*\|_{\mathcal{A}}$.

Examples

- Independent GUE random matrices (Haagerup, Thorbjørnsen 2005).
- Independent GOE and GSE random matrices (Schultz 2005).
- Some independent Wigner matrices (Capitaine, Donati-Martin 2007; Anderson 2013; Belinschi, Capitaine 2017)
- Independent Wishart matrices (Capitaine, Donati-Martin 2007)
- Independent Haar unitary matrices (Collins, Male 2014)

Rational functions in strongly convergent random matrices

Proposition (Y. 2017)

Suppose that a sequence of tuples $X^N = (X_1^N, \cdots, x_m^N) \in M_N(\mathbb{C})^m$ converges strongly in distribution to a tuple $x = (x_1, \cdots, x_m)$ of operators and the tuple (x, x^*) lies in the domain of a rational function $r \in \mathscr{R}$, i.e., $r(x, x^*)$ is well-defined as a bounded operator. Then we have

- $(X^N, (X^N)^*)$ lies in the domain of r eventually;
- the convergence of trace and norms, i.e.,

$$\lim_{N\to\infty} \mathrm{tr}_N(r(X^N, (X^N)^*)) = \tau(r(x, x^*)),$$

$$\lim_{n\to\infty} \left\| r(x^{(n)}, (x^{(n)})^*) \right\| = \left\| r(x, x^*) \right\|.$$

Basic idea: boundedness of inverse + strong convergence $\implies$ least singular values stay away from zero

A naive example: Let x^{-1} be well-defined as a bounded operator, then

$$\min\{\mathrm{Spec}(xx^*)\} = \| \|xx^*\| - xx^* \| > 0$$

and $p(x, x^*) = \|xx^*\| - xx^*$ is a polynomial.
Now suppose that a sequence of matrices X^N strongly converges to x, then we can show that

$$\lim_{N\to\infty} \| \|X^N(X^N)^*\| - X^N(X^N)^* \| = \| \|xx^*\| - xx^* \| > 0,$$

so X^N is invertible eventually. Moreover, the norm converges,

$$\|x^{-1}\| = \| \|xx^*\| - xx^* \|^{-\frac{1}{2}} = \lim_{N\to\infty} \|(X^N)^{-1}\|.$$

Rational functions in strongly convergent random matrices

Recursive structure of rational functions

If X^N strongly converges to $x \in \mathcal{A}$ and x lies in the domain of $r \in \mathcal{R}_1 =< \mathcal{P} \cup \mathcal{P}^{-1} >$, then the above argument shows that X^N lies in the domain of r eventually. Therefore, roughly speaking, this means that $\left(r\left(x\right), r(X^N), \cdots\right)$ belongs to the C*-algebra $\mathcal{A} \times \prod_{n \geqslant N} M_n\left(\mathbb{C}\right)$ equipped with the sup norm, which can allow us to find a polynomial p to approximate r over this C*-algebra. And thus we have the norm convergence for $r \in \mathcal{R}_1$.

So we can repeat the above argument by replacing strong convergence with norm convergence for all possible $\mathcal{R}_1$-functions, to go to $\mathcal{R}_2$-functions, and so on.

Norm convergence $\Longrightarrow$ trace convergence

$$\begin{aligned} |\mathrm{tr}_N(r^N) - \tau(r)| &\leqslant |\mathrm{tr}_N(r^N - p^N)| + |\mathrm{tr}_N(p^N) - \tau(p)| + |\tau(p - r)| \\ &\leqslant \|r^N - p^N\| + |\mathrm{tr}_N(p^N) - \tau(p)| + \|\tau(p - r\|. \end{aligned}$$

Moment convergence is not stable w.r.t inversing

Example

Let $X^{(n)} \in M_n(\mathbb{C})$ be a sequence of matrices that strongly converges to x. And we assume that x is invertible, then $X^{(n)}$ is invertible eventually, and

$$\lim_{n\to\infty} \mathrm{tr}_n((X^{(n)})^{-1}) = \tau(x^{-1}).$$

Now put

$$Y^{(n+1)} = \begin{pmatrix} \frac{1}{n+1} & 0 \\ 0 & X^{(n)} \end{pmatrix} \in M_{n+1}(\mathbb{C}),$$

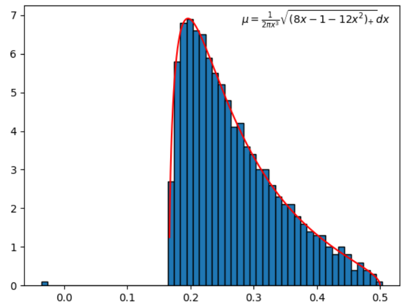
then it is clear that $Y^{(n)}$ also converges in moments to x and $Y^{(n)}$ is invertible as $X^{(n)}$ is invertible eventually. However, we can see that

$$\lim_{n\to\infty} \mathrm{tr}_n((Y^{(n)})^{-1}) = 1 + \tau(x^{-1}).$$

Random matrices with outliers

Reminder: This is also a "nice" random matrix

$(4 - s)^{-1}$



Zero divisors and atoms

Definition

Let $(\mathcal{N}, \tau)$ be a W^*-probability space, i.e., $\mathcal{N} \subseteq B(H)$ is a finite von Neumann algebra with faithful trace τ s.t. $\tau(1) = 1$. For an element $x \in \mathcal{N}$, if there is some non-zero element $y \in \mathcal{N}$ s.t. $xy = 0$ or $yx = 0$, then we say x has a zero divisor y over $\mathcal{N}$.

Definition

For a distribution μ on the real line $\mathbb{R}$, if $\mu(\alpha) > 0$, then we say μ has an atom at α.

Remark

Let μ be the measure given by a self-adjoint element $x \in \mathcal{N}$ s.t. $\tau(x^n) = \int z^n d\mu(z)$. Then, by the spectral theorem, μ has no atoms iff there is no zero divisor for $x - \alpha$, $\forall \alpha \in \mathbb{R}$.

Zero divisors and atoms

Theorem (Charlesworth, Shlyakhtenko; Mai, Speicher, Weber 2015)

Let $(\mathcal{N}, \tau)$ be a W^*-probability space and $(x_1, \cdots, x_n)$ a tuple of self-adjoint elements. If $x_1, \cdots, x_n$ have the maximal non-microstate free entropy dimension, then for any non-trivial polynomial p, $p(x_1, \cdots, x_n)$ has no zero divisors.

Remarks

- In particular, if $x_1, \cdots, x_n$ are free semicirculars, then $p(x_1, \cdots, x_n)$ has no zero divisors for any polynomial p.
- If p is self-adjoint, then the distribution of $p(x_1, \cdots, x_n)$ has no atoms.
- Replacing the polynomials by rational functions, does this result still hold?

Measurable operators and their invertibility

Definition

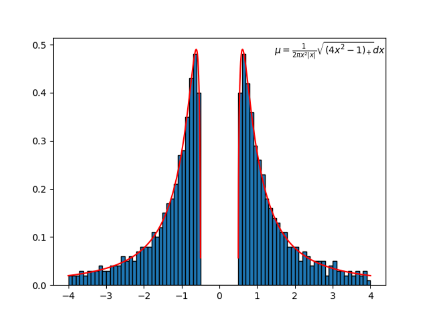
Let $(\mathcal{N}, \tau)$ be a W^*-probability space. A closed densely defined operator x on H is said to be affiliated with $\mathcal{N}$ if $ux = xu$ for any unitary u in the commutant $\mathcal{N}'$ of $\mathcal{N}$. We denote by $L_0(\mathcal{N})$ the family of all affiliated operators with $\mathcal{N}$. An element in $L_0(\mathcal{N})$ is also called measurable operator.

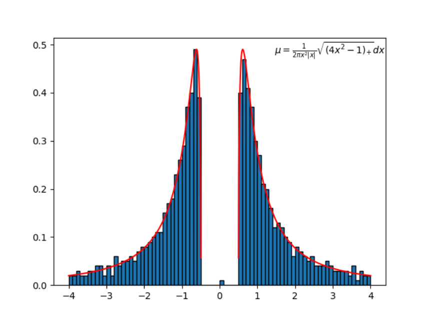
Remarks

- In particular, if $\mathcal{N} \cong L_\infty(\Omega, \mu)$, then $L_0(\mathcal{N})$ is the $*$-algebra of measurable functions.
- $L_0(\mathcal{N})$ is a $*$-algebra, which is also stably finite.
- An element $x \in L_0(\mathcal{N})$ is invertible if and only if x has no zero divisors.

Inverse of a semi-circular operator

A semi-circular is invertible as a measurable operator





Zero divisors and invertibility

Theorem (Charlesworth, Shlyakhtenko; Mai, Speicher, Weber 2015)

Let $(\mathcal{N}, \tau)$ be a W^*-probability space and $L_0(\mathcal{N})$ the $*$-algebra of measurable operators affiliated with $\mathcal{N}$. If $x_1, \cdots, x_n$ are self-adjoint elements which have the maximal non-microstate free entropy dimension, then for any non-trivial polynomial p, $p(x_1, \cdots, x_n)$ is invertible in $L_0(\mathcal{N})$.

For a non-trivial rational function r, is $r(x_1, \cdots, x_n)$ invertible, too?

In one variable case, yes! It is simply because every rational function r has a representation of form pq^{-1}.

Zero divisors and invertibility

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For a non-trivial rational function r, is $r(x_1, \cdots, x_n)$ invertible, too?

In multi-variables case, it could be true, but we can't deduce directly from the invertibility of polynomials any more!
For example (K. Dykema, J. Pascoe): let x and y be two free semicirculars,

$$a := x^2,\ b := xyx,\ c := xy^2x,$$

then $p(a, b, c)$ is invertible for each polynomial but they satisfy a rational relation $ba^{-1}b = c$. This means that rational function $r = z_2 z_1^{-1} z_2 - z_3$ is not invertible, or has every element in $\mathcal{N}$ as its zero divisor.

Strong Atiyah property

Definition

Let $(\mathcal{N}, \tau)$ be a W^*-probability space and $x_1, \cdots, x_n \in \mathcal{N}$. If for any matrix A of polynomials, we have

$$\dim \ker A(x_1, \cdots, x_n) = \mathrm{Tr}_N \otimes \tau(p_{\ker A(x_1, \cdots, x_n)}) \in \mathbb{Z} \cap [0, N],$$

then we say $(x_1, \cdots, x_n)$ has the strong Atiyah property.

Examples

- (Linnell 1993) Let $u_1, \cdots, u_n$ be the generators of group von Neumann algebra $L(\mathbb{C}\mathbb{F}_n)$, then $(u_i, u_i^*)_{i=1}^n$, or simply, $(u_i)_{i=1}^n$ has strong Atiyah property.
- (Shlyakhtenko, Skoufranis 2013) A tuple of freely independent normal operators has strong Atiyah property if the distribution of each element in the tuple has no atoms. In particular, a tuple of free semicirculars $(s_1, \cdots, s_n)$ also has strong Atiyah property.

Strong Atiyah property and rational closure

Lemma (Linnell 1993)

If $(x_1, \cdots, x_n)$ has Strong Atiyah property, then there is a subalgebra R of $L_0(\mathcal{N})$, called rational closure, s.t.

- *every element of $\mathcal{R}$ is given as an evaluation of some rational function $r(x_1, \cdots, x_n)$ and $\mathbb{C} < x_1, \cdots, x_n > \subseteq \mathcal{R} \subseteq L_0(\mathcal{N})$,*
- *R is a division algebra, i.e., each non-zero element in R is invertible.*

So we know that

- Free semicirculars/unitaries $\Longrightarrow$ Strong Atiyah property $\Longrightarrow$ A lot of rational functions are invertible
- Free semicirculars $\Longrightarrow$ Maximal free entropy dimension $\Longrightarrow$ All non-zero polynomials are invertible

Embedding of Free field

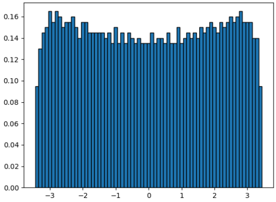
Proposition (Linnell 1993)

The rational closure of $\mathbb{CF}_n$ in the $*$-algebra of measurable operators $L_0(\mathrm{vN}(\mathbb{CF}_n))$ is isomorphic to the universal field of fractions for $\mathbb{CF}_n$.

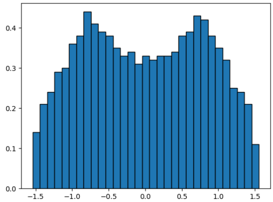
Corollary

Let $X^N = (X_1^N, \cdots, x_m^N)$ be a sequence of independent Haar unitary random matrices, then for any non-trivial rational function r s.t. $r(X^N)$ is bounded self-adjoint operator, then its limiting distribution has no atom.

$U + V + U^{-1} + V^{-1}$



$(2 - U)^{-1}V + V^{-1}(2 - U^{-1})^{-1}$



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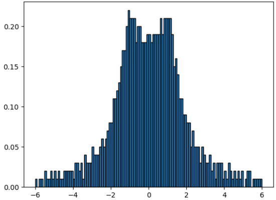
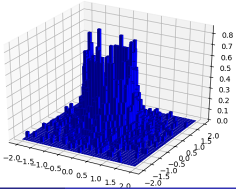
Conjecture

Let $X^N = (X_1^N, \cdots, x_m^N)$ be a sequence of independent Haar unitary random matrices, then for any non-trivial rational function r, the limiting measure of $r(X^N)$ has no atom.

$(1-U)^{-1}V + V^{-1}(1-U^{-1})^{-1}$ $\qquad$ $(1-U)^{-1}V$





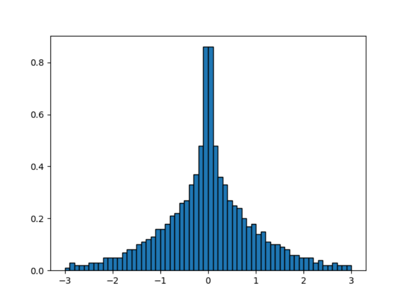
Embedding of Free field

And we want to know that

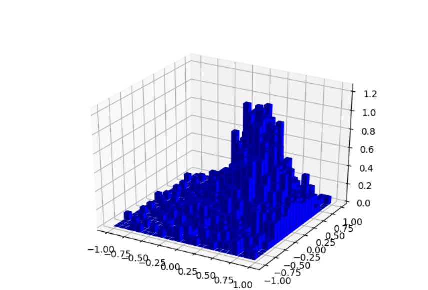
- Free semicirculars $\implies$ All non-zero rational functions are invertible?
- Maximal free entropy dimension $\implies$ All non-zero rational functions are invertible? (Ongoing project with Tobias Mai)
- Maximal free entropy dimension $\implies$ Strong Atiyah property?

Let X, Y be GUE matrices

$X(X+Y)^{-1}X$



$X(X+Y)^{-1}$



Merci!